Purification of Mixed State with Closed Timelike Curve is not Possible

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In ordinary quantum theory any mixed state can be purified in an enlarged Hilbert space by bringing an ancillary system. The purified state does not depend on the state of any extraneous system with which the mixed state is going to interact and on the physical interaction. Here, we prove that it is not possible to purify a mixed state that traverses a closed time like curve (CTC) and allowed to interact in a consistent way with a causality-respecting (CR) quantum system in the same manner. Thus, in general for arbitrary interactions between CR and CTC systems there is no universal 'Church of the larger Hilbert space' for mixed states with CTC. This shows that in quantum theory with CTCs there can exist 'proper' and 'improper' mixtures.

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For many years physicists believed that the existence of closed timelike curve (CTC) [1–5] is only a theoretical possibility rather than a feasibility. A closed time like curve typically connects back on itself, for example, in the presence of a spacetime wormhole that could link a future spacetime point with a past spacetime point. However, there have been criticisms to the existence of CTCs and the "grandfather paradox" is one. But Deutsch proposed a computational model of quantum systems in the presence of CTCs and resolved this paradox by presenting a method for finding self-consistent solutions of CTC interactions in quantum theory [6]. He has also investigated various quantum mechanical effects that can occur on and near the closed timelike curves, including the possibility of violation of the correspondence principle and unitarity. This has opened up a new area of research in recent times. In quantum information theory some authors have assumed that CTCs exist and examined the consequences of this assumption for quantum computation [7–9]. Several authors have come up with the idea that access to CTCs could enhance the computational power. For example, in reference [7] it was proposed that access to a CTC would factor composite numbers efficiently with the help of a classical computer only. This clearly gave strong indications that the CTC-assisted computers may be more powerful. The author in reference [8] has moved one step forward and argued that a CTC-assisted quantum computer would be able to solve NP-complete problems, which is an impossibility for a quantum computer alone. Another result about computational complexity is that the power of a polynomial time bounded computer in classical or quantum world assisted by CTCs can be solved in a space polynomial in the problem size (PSPACE) but potentially exponential time [9].

In an interesting work Brun *et al* [10] have shown how a party with access to CTCs can perfectly distinguish nonorthogonal quantum states. The result came as a shock to the community as it put fundamental quantum cryptographic protocols (like BB84 [11] protocol) at stake. This is so because the encrypting scheme that depends on the principles of quantum theory can be broken by allowing the quantum message to interact with another quantum system that travel through a CTC region. At the same time this raises several questions about the nature of the quantum world with closed timelike curves.

However, Bennett *et al* have a completely opposite view to this proposal [12]. They studied the power of closed timelike curves (CTCs) for the tasks of distinguishing nonorthogonal input states and speeding up otherwise hard computations. They showed that when a CTC-assisted computer is feed up with a labelled mixture of states as an input it can do no better than it would have done without CTC assistance. The conclusions drawn in [10] are based on considering only fixed input states which are pure. These conclusions do not hold for general case where inputs consist of mixed states. They resolved the contradiction between their claim and previous results by arguing that as the CTC-assisted evolution is not linear, the output of such computation on a mixture of inputs is not a convex combination of its output on the pure components of the mixture. Recently, [13], it has been argued that by keeping track of the information flow of quantum system during the interaction with CTCs one can have a better picture of the Deutsch model and this also supports the notion that CTC assisted party can distinguish non-orthogonal quantum states as reported in [10]. This once again suggests that the whole situation is worth rethinking about the true power of CTC world as there is no direct experimental evidence whether having access to the CTC qubit or nonlinearity do provide any help in this regard. It may be worth mentioning another strange feature of the Deutsch model, namely, that the dynamical evolution through the region containing CTCs become discontinuous function of the input state [14].

Now coming back to Deutsch formalism [6], it involves a unitary interaction between a causality-respecting (CR) quantum system with another system that is having a world line in the region of a closed timelike curve (CTC). It is assumed that these states are quite similar to the density matrices that we have in standard quantum mechanics. Before the interaction the combined state of the CR system and CTC system is in a product state $\rho_{\rm CR}\otimes\rho_{\rm CTC}$. The unitary interaction between the CR and CTC systems transforms the initial composite state as

$$\rho_{\rm CR}\otimes\rho_{\rm CTC}\to U(\rho_{\rm CR}\otimes\rho_{\rm CTC})U^{\dagger}. \qquad (1)$$

For each initial mixed state $\rho_{\rm CR}$ of the CR system, there do exists a CTC system $\rho_{\rm CTC}$ such that after the interaction we must have

$$\mathrm{Tr}_{\rm CR}(U\rho_{\rm CR}\otimes\rho_{\rm CTC}U^{\dagger}) = \rho_{\rm CTC}. \qquad (2)$$

This is the self-consistency condition in the Deutsch model. Since the time travelled system follows a CTC, travelling back in time will make the input density matrix equal to its output density matrix. Mathematically, the solution to this equation is a fixed point (there may be multiple fixed points, in which case one has to take the maximum-entropy mixture of them) [6]. The final state of the CR quantum system is then defined as

$$\mathrm{Tr}_{\mathrm{CTC}}(U\rho_{\mathrm{CR}}\otimes\rho_{\mathrm{CTC}}U^{\dagger})=\rho_{\mathrm{CR}}^{'}. \tag{3}$$

In a nutshell these are the two basic conditions which actually govern the entire unitary dynamics of the CTC and CR quantum systems. Here, one should note that ρ_{CTC} depends nonlinearly on ρ_{CR} and hence the output of the CR system is a nonlinear function of the input density matrix [15].

In this paper we ask what should be the nature of the density operator of a quantum system that traverses a closed time like curve. Is it similar in all respect to causality-respecting quantum system? In particular, we ask can we always purify ρ_{CTC} that interacts with a CR system indepedent of anything else? Quit surprisingly, we show that there does not exist general purification for a mixed state density operator of a CTC quantum system in the Deutsch model for arbitrary ρ_{CR} and U. In other words, given a CTC system in the Deutsch formalism which is in a mixed state, if it has to interact with arbitrary CR system and subject to arbitrary interaction, then that cannot be viewed as a subsystem of a universal pure entangled state in an enlarged Hilbert space of the composite CTC system (up to local unitary freedom). Since, in quantum information theory purification of mixed states finds many important applications, we believe that our result have deep implications for quantum theory in the presence of CTCs. Recently, it was shown that in general probabilistic theories that allow for purification one can obtain most of the quantum formalism and principles of quantum information processing [16]. This may provide a strong motivation to our result, showing that closed timelike curves exclude the universal purification implies that they exclude a central piece of quantum mechanics.

In quantum theory the density operator $\rho\in\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the set of density operators on a given Hilbert space $\mathcal{H}$. It is a Hermitian, positive and trace class operator with $\mathrm{Tr}(\rho)=1$. In the ordinary quantum world there is no distinction between a 'proper mixture' and an 'improper mixture'. The *proper mixture* ρ is a convex combination of subensembles of pure states $|\psi_k\rangle$, each occurring with a probability p_k. Thus, ρ can be expressed as $\rho=\sum_k p_k|\psi_k\rangle\langle\psi_k|$ and this bears the ignorance interpretation. The above decomposition is not unique and can be expressed in infinitely many ways as a convex sum of distinct, but not necessarily orthogonal projectors. This notion was introduced by Von Neumann when we do not know the exact preparation procedure in describing the physical system. The *improper mixture* corresponds to the result of tracing out of a pure projector of a composite system (system plus ancilla) $S+A$ such that $\rho_S=\mathrm{Tr}_A(|\Psi\rangle_{\mathrm{SA}}\langle\Psi|)$, where $|\Psi\rangle_{\mathrm{SA}}=\sum_k\sqrt{p_k}|\psi_k\rangle_S|\phi_k\rangle_A$. However, there is no way to differentiate a proper mixture from improper one. This is so, because, given any density matrix we can always purify it in an enlarged Hilbert space (in infinite number of ways). In fact, it has been argued that in the usual quantum theory all the density matrices can be regarded as 'improper mixture' [17]. Most importantly, the pure entangled state does not depend on the state of other extraneous system with which ρ_{S} is going to interact nor it depends on the interaction that might take place between the system and anything else. However, this is not the case for CTC quantum systems.

Theorem I: Let $\rho_{\mathrm{CTC}}=\sum_k p_k|k\rangle\langle k|$ be the spectral decomposition of the CTC mixed state that interacts with a CR quantum system in a pure state. For arbitrary U on $\mathcal{H}_{\mathrm{CR}}\otimes\mathcal{H}_{\mathrm{CTC}}$ and $|\psi\rangle_{\mathrm{CR}}\langle\psi|$ on $\mathcal{H}_{\mathrm{CR}}$ if the CTC density matrix satsifies the consistency condition $\rho_{\mathrm{CTC}}=\mathrm{Tr}_{\mathrm{CR}}(U\rho_{\mathrm{CR}}\otimes\rho_{\mathrm{CTC}}U^{\dagger})$ then there does not exist general purification for ρ_{CTC}, such that $\rho_{\mathrm{CTC}}=\mathrm{Tr}_{\mathrm{CTC}'}[|\Phi\rangle\langle\Phi|]$, where $|\Phi\rangle_{\mathrm{CTC,CTC}'}=\sum_k\sqrt{p_k}|k\rangle_{\mathrm{CTC}}|b_k\rangle_{\mathrm{CTC}'}$ for all ρ_{CR} and U.

Proof: We start with two quantum systems as inputs to the unitary evolution, where one system is from the causality respecting (CR) region and another from the causality violating region. Let these density matrices are given by $\rho_{\mathrm{CR}}=|\psi\rangle_{\mathrm{CR}}\langle\psi|$ and $\rho_{\mathrm{CTC}}=\sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|$. Now, they undergo unitary evolution in the Deutsch model according to (1). Let us assume that the purification is possible for the CTC system. Then, it can be thought of as a part of a pure entangled state of a composite system in the Hilbert space $\mathcal{H}_{\mathrm{CTC}}\otimes\mathcal{H}_{\mathrm{CTC}'}$. This pure entangled state can be written in the Schmidt decomposition form as (up to local unitary in the Hilbert space of CTC')

$$|\Phi\rangle_{\mathrm{CTC,CTC}'}=\sum_k\sqrt{p_k}|k\rangle_{\mathrm{CTC}}|b_k\rangle_{\mathrm{CTC}'} \tag{4}$$

with $\rho_{\mathrm{CTC}}=\mathrm{Tr}_{\mathrm{CTC}'}[|\Phi\rangle\langle\Phi|]$. Here, p_k's are the Schmidt coefficients, $|k\rangle$'s are orthonormal basis for the CTC Hilbert space and $|b_k\rangle$ are orthonormal basis for the CTC' Hilbert space. In the pure state picture the unitary evolution is equivalent to

$$\begin{aligned}&|\psi\rangle_{\mathrm{CR}}\langle\psi|\otimes|\Phi\rangle_{\mathrm{CTC,CTC}'}\langle\Phi|\rightarrow\\&U\otimes I(|\psi\rangle_{\mathrm{CR}}\langle\psi|\otimes|\Phi\rangle_{\mathrm{CTC,CTC}'}\langle\Phi|)U^{\dagger}\otimes I,\end{aligned} \tag{5}$$

where U acts on the Hilbert space of CR and CTC and I acts on CTC'. If the purification holds then after the evolution the combined state of the CR and CTC system is given by

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{CTC}'}[U\otimes I(|\psi\rangle_{\mathrm{CR}}\langle\psi|\otimes|\Phi\rangle_{\mathrm{CTC,CTC}'}\langle\Phi|)U^{\dagger}\otimes I]=\\&\qquad=\sum_k p_kU(|\psi\rangle_{\mathrm{CR}}\langle\psi|\otimes|k\rangle_{\mathrm{CTC}}\langle k|)U^{\dagger}.\end{aligned} \tag{6}$$

From (1) and (6) we have

$$U(|\psi\rangle_{\rm CR}\langle\psi| \otimes \sum_k p_k |k\rangle_{\rm CTC}\langle k|)U^\dagger = \sum_k p_k U(|\psi\rangle_{\rm CR}\langle\psi| \otimes |k\rangle_{\rm CTC}\langle k|)U^\dagger. \quad (7)$$

Now taking the partial trace over the CR system and using the consistency condition we have

$$\sum_k p_k |k\rangle_{\rm CTC}\langle k| = {\rm Tr}_{\rm CR}[\sum_k p_k U(|\psi\rangle_{\rm CR}\langle\psi| \otimes |k\rangle_{\rm CTC}\langle k|)U^\dagger]. \quad (8)$$

Now the action of the unitary operator on the CR system and the orthonormal basis of the CTC system can be written in the Schmidt decomposition form as

$$U(|\psi\rangle_{\rm CR}|k\rangle_{\rm CTC}) = \sum_m \sqrt{f_m^k(\psi)}|a_m^k(\psi)\rangle_{\rm CR}|u_m(k)\rangle_{\rm CTC}, \quad (9)$$

where $f_m^k(\psi)$'s are the Schmidt coefficients with $\sum_m f_m^k(\psi) = 1$ for all k, $|a_m^k(\psi)\rangle_{\rm CR}$ and $|u_m(k)\rangle_{\rm CTC}$ are orthonormal Schmidt bases for the CR system and CTC system, respectively. Therefore, from (8) we have

$$\sum_k p_k |k\rangle\langle k| = \sum_{km} p_k f_m^k(\psi)|u_m(k)\rangle\langle u_m(k)|. \quad (10)$$

From (10), the eigenvalues of $\rho_{\rm CTC}$ are given by

$$p_n = \sum_{km} p_k f_m^k(\psi)|\langle n|u_m(k)\rangle|^2 \quad (11)$$

which depends on $\rho_{\rm CR}$ and U. This is a recursive equation. However, the spectrum of $\rho_{\rm CTC'}$ is independent of $\rho_{\rm CR}$ and U (as it does not interact with anything and in principle can be far away from the region where CR and CTC systems interact). Since we must have the equal spectrum for any pure bipartite entangled state, i.e., Spec($\rho_{\rm CTC}$) = Spec($\rho_{\rm CTC'}$), the purification assumption and consistency condition violate this. Thus, for any non-trivial interaction between the CR and the CTC to happen, our assumption that the purification is possible for $\rho_{\rm CTC}$ must be wrong. Hence, there is no purification of a CTC mixed state for arbitrary CR system and arbitrary interaction.

Our proof also rules out the possibility that a CTC mixed state can be purified by attaching a CR system as part of the purification. Even if we imagine that CTC can be purified with a CR system, then similar arguments will go through and we will get a contradiction. Thus, *there is no universal 'Church of the larger Hilbert space' (either in CR or CTC world) where a CTC mixed state can be purified* if it has to undergo arbitrary interaction with arbitrary CR system in a non-trivial way. It will be interesting to see if the no-purification theorem holds in other non-linear quantum theories [18].

Does our result rule out the existence of 'improper' mixture? The answer is no. If already we have a composite CTC system in a pure entangled state, then the subsystems will be in a mixed state and that will be a 'improper mixture'. As long as the part of the entangled state of a CTC system does not interact with a CR system in a pure state, the existence of 'improper mixture' is allowed. One may aruge that for a fixed known $\rho_{\rm CR}$ and U one can purify $\rho_{\rm CTC}$. But then the pure entangled state depends on $\rho_{\rm CR}$ and U, i.e., $|\Phi\rangle = |\Phi(\psi, U)\rangle$. This is in sharp distinction to the purification in ordinary quantum theory: If we have two systems (say) with density matrices ρ and $\rho_{\rm S}$ and they interact via $\rho \otimes \rho_{\rm s} \rightarrow U(\rho \otimes \rho_{\rm s})U^\dagger$, then we can always purifiy $\rho_{\rm S}$ such that $\rho_S = {\rm Tr}_A(|\Psi\rangle_{\rm SA}\langle\Psi|)$, where $|\Psi\rangle_{\rm SA}$ is the purified state and it does not depend on ρ and U. The purification of a density matrix in ordinary quantum theory is universal. However, in the case of CTC quantum theory it is not so. Also, in the case of CTC quantum theory, it is not clear why the eigenvalues of $\rho_{\rm CTC'}$ which may be far away depends on the $\rho_{\rm CR}$ and U which it has not seen. There seems to be a problem with the locality principle in quantum theory with CTCs. Nevertheless, there can be a sepcial unitary and $\rho_{\rm CTC}$ when the puification may be possible. For example, when $f_m(k) = 1/d$ for all k, m and $p_n = 1/d$ for all n (the CTC density matrix is a random mixture), then Spec($\rho_{\rm CTC}$) = Spec($\rho_{\rm CTC'}$) holds and purification of CTC density matrix is possible for arbitrary $\rho_{\rm CR}$.

Therefore, in quantum world where CTCs exist we do have two kind of mixtures. Even though practitioners of standard quantum theory do not think that there should be a difference between 'proper' and 'improper' mixtures, there are some physicists and philosophers [19, 20] who seem to believe that these two are distinct. They sometimes also discuss it in terms of objective versus subjective probabilities. The purification not being possible for the CTC system would imply that there could be objective probabilities after all and presumably would say something about the supposed reduction of classical probabilities to pure quantum evolution that the many worlds people would like to believe.

Now, one might think that if the state of a CR system and a CTC system is already in a pure entangled state (say) $|\Phi\rangle_{\rm CR,CTC} = \sum_k \sqrt{p_k}|a_k\rangle_{\rm CR}|k\rangle_{\rm CTC}$, then by tracing out the CR system the state of the CTC subsystem will be in a mixed state. Now the question is can we always create such an entangled state between a CR and CTC system using any unitary? One can argue that this may not be the case. Because if a CR and a CTC system is in a pure entangled state then at some point of time they must have interacted. Once they interact the CTC system has to satisfy the consistency condition of Deutsch. One can check that this will lead to contradictions. Suppose we start with a CR system and a CTC system in pure product states, i.e., $\rho_{\rm CR} \otimes \rho_{\rm CTC} = |\psi\rangle_{\rm CR}\langle\psi| \otimes |\phi\rangle_{\rm CTC}\langle\phi|$ for some $|\psi\rangle \in \mathcal{H}_{\rm CR}$ and for some $|\phi\rangle \in \mathcal{H}_{\rm CTC}$. Then after the evolution the state of the CR and CTC system is $|\Phi\rangle\langle\Phi|$ with $\rho_{\rm CTC} = \sum_k p_k|k\rangle\langle k|$. This shows that the consistency condition is not satisfied. Similarly, if we start with the initial $\rho_{\rm CR}$ as a pure state and the initial $\rho_{\rm CTC}$ as a mixed state, then the question is how can an overall mixed state (CR and CTC) evolve to a pure state $|\Phi\rangle_{\rm CR,CTC}$ via unitary transformation.

In either case we reach a contradiction.

So the question is how can one create entanglement between a CR system and a CTC system? It is possible to create entanglement between a CR system and a CTC system by first creating entanglement between two CR systems (in our world) and then swapping half of the CR subsystem with a CTC system whose density matrix is same as the reduced density matrix of the CR system. To illustrate this clearly, let us consider two causality-respecting quantum systems in a pure entangled state $|\Psi\rangle_{\mathrm{CR,CR'}} = \sum_k \sqrt{p_k}|a_k\rangle_{\mathrm{CR}}|k\rangle_{\mathrm{CR'}}$. Let the initial state of $\rho_{\mathrm{CTC}} = \sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|$. Let half of the causality-respecting system $\mathrm{CR'}$ and the CTC system interact where the unitary operation is the swap operation. After this interaction we will find that the state of the other half of the CR system and the CTC system is actually in an pure entangled state $|\Psi\rangle_{\mathrm{CR,CTC}} = \sum_k \sqrt{p_k}|a_k\rangle_{\mathrm{CR}}|k\rangle_{\mathrm{CTC}}$ with the fixed point solution $\rho_{\mathrm{CTC}} = \sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|$. Now one may wonder, since the older version of the CTC system is same as the younger version of the CTC system and the younger version of the CTC state which was in a mixed state is actually now part of a pure entangled state how does it bypass the no-purification theorem? Below we answer this question.

Theorem II: Let $\rho_{\mathrm{CR'}} = \sum_i \lambda_i|b_i\rangle_{\mathrm{CR'}}\langle b_i|$ and $\rho_{\mathrm{CTC}} = \sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|$ are the density matrices of the causality-respecting system and CTC system, with their respective spectral decompositions. Let them interact via an arbitrary unitary operator U that acts on $\mathcal{H}_{\mathrm{CR'}} \otimes \mathcal{H}_{\mathrm{CTC}}$. If ρ_{CTC} satisfies the consistency condition, then it cannot be purified for arbitrary $\rho_{\mathrm{CR'}}$ and U such that $\rho_{\mathrm{CTC}} = \mathrm{Tr}_{\mathrm{CTC'}}(|\Phi\rangle_{\mathrm{CTC,CTC'}}\langle\Phi|)$ with $|\Phi\rangle_{\mathrm{CTC,CTC'}} = \sum_k \sqrt{p_k}|k\rangle_{\mathrm{CTC}}|d_k\rangle_{\mathrm{CTC'}}$.

Proof: Let the causality-respecting system that interacts with CTC system is part of a pure entangled states, i.e., $\rho_{\mathrm{CR'}} = \mathrm{Tr}_{\mathrm{CR}}(|\Psi\rangle_{\mathrm{CR,CR'}}\langle\Psi|)$ with $|\Psi\rangle_{\mathrm{CR,CR'}} = \sum_i \sqrt{\lambda_i}|a_i\rangle_{\mathrm{CR}}|b_i\rangle_{\mathrm{CR'}}$. Next, assume that we can purify the CTC system, i.e., $\rho_{\mathrm{CTC}} = \mathrm{Tr}_{\mathrm{CTC'}}(|\Phi\rangle_{\mathrm{CTC,CTC'}}\langle\Phi|)$ with $|\Phi\rangle_{\mathrm{CTC,CTC'}} = \sum_k \sqrt{p_k}|k\rangle_{\mathrm{CTC}}|d_k\rangle_{\mathrm{CTC'}}$. In the pure state picture, the Deutsch unitary evolution between the causality-respecting system and the chronology violating system can have the following transformation

$$|\Psi\rangle_{\mathrm{CR,CR'}}|\Phi\rangle_{\mathrm{CTC,CTC'}} \to (I\otimes U\otimes I)|\Psi\rangle_{\mathrm{CR,CR'}}|\Phi\rangle_{\mathrm{CTC,CTC'}}. \tag{12}$$

Thus, if the purification is possible then we have

$$\begin{aligned} & U[\sum_i \lambda_i|b_i\rangle_{\mathrm{CR'}}\langle b_i| \otimes \sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|]U^\dagger \\ & = \sum_{ik} \lambda_i p_k U(|b_i\rangle_{\mathrm{CR'}}\langle b_i| \otimes |k\rangle_{\mathrm{CTC}}\langle k|))U^\dagger. \end{aligned} \tag{13}$$

Let us define the action of the unitary operator on the basis of $\mathcal{H}_{\mathrm{CR'}}$ and $\mathcal{H}_{\mathrm{CTC}}$ as

$$U(|b_i\rangle_{\mathrm{CR'}}|k\rangle_{\mathrm{CTC}}) = \sum_m \sqrt{g_m(i,k)}|B^k_m(i)\rangle_{\mathrm{CR'}}|C^i_m(k)\rangle_{\mathrm{CTC}}, \tag{14}$$

where $g_m(i,k)$'s are the Schmidt coefficients with $\sum_m g_m(i,k) = 1$ for all (i,k), $|B^k_m(i)\rangle_{\mathrm{CR'}}$ and $|C^i_m(k)\rangle_{\mathrm{CTC}}$ are the Schmidt bases in their respective Hilbert spaces. If we trace out $\mathrm{CR'}$ and use the kinematic consistency condition we have

$$\sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k| = \sum_k p_k \sum_{im} \lambda_i g_m(i,k)|B^k_m(i)\rangle_{\mathrm{CTC}}\langle B^k_m(i)|. \tag{15}$$

From (15), we have

$$p_n = \sum_{ikm} p_k\lambda_i g_m(i,k)|\langle n|B^k_m(i)\rangle|^2. \tag{16}$$

Thus, the egienvalues of ρ_{CTC} depends on $\rho_{\mathrm{CR'}}$ and U whereas the egienvalues of $\rho_{\mathrm{CTC'}}$ are independent of the later. Therefore, if purification is possible for arbitrary $\rho_{\mathrm{CR'}}$ and U then the equal spectrum condition is violated. Hence, the proof.

However, there are some special unitaries and CTC density matrix when purification can hold. For example, when the $\mathrm{CR'}$ system and the CTC system are subject to swap operation, i.e., $U = U_{\mathrm{SWAP}}$ and $\rho_{\mathrm{CTC}} = \rho_{\mathrm{CR'}}$. In this process, entanglement is created between the CR and the CTC systems, and between the $\mathrm{CR'}$ and the $\mathrm{CTC'}$ systems. Also, general purification can hold when $g_m(i,k)$ and p_n are equal (say $1/d$, i.e. for a random CTC density matrix) for all i,k,m,n.

In the Deutsch formalism, the mixed state ρ_{CTC} is constrained by (2). Also this is in the form of a product state with the causality respecting quantum system. This allows the state of the system to evolve from a pure state to a mixed state, which is forbidden in the standard quantum mechanics. One may be tempted to have an entangled pure state picture as suggested by Deutsch where one takes into consideration of the many-worlds interpretation [6]. That is the CTC subsystem in our world can be regarded as being entangled with the CTC and CR subsystems in other world. However, our result rules out this possibility (as the interacting CTC cannot be correlated with another world in a universal manner).

To conclude, we have proved a no-purification theorem for quantum systems in mixed states that traverse closed time like curves. It is shown that given a CTC system in a mixed state, if it has to evolve via Deutsch unitary gate and satisfy the kinematic consistency condition for all ρ_{CR} and all U then it cannot be regarded as a subsystem of a universal pure entangled state. Thus, in general, there is no universal ‘Church of the larger Hilbert space’ for CTC quantum system. However, as long as a CTC system does not interact with a CR system in a nontrivial way ‘improper’ mixture may exist. One may argue that for a fixed known ρ_{CR} and U one can purify a CTC density matrix, but the physical meaning of such a context dependent purification is not clear. In particular, if the other half of the CTC system is far away, then why should that depend on what is going on at a remote location. Therefore, in quantum theory with closed time like curves, there can be two kinds of mixtures, namely, ‘proper’ and ‘improper’ mixtures.

This also reveals the true nature of the density operators in quantum theory with CTCs. In the absence of CTCs, there is no way to distinguish them. If CTC can help in distinguishing 'proper' from 'improper' mixture then it may lead to signalling. How exactly this can happen will be reported in future. In essence our result brings out a very fundamental and important difference between the density matrix of the CR system and the CTC system. It is likely that the nature of entanglement between causality-respecting system and CTC system may be different. We hope that our result will add new insights to quantum information theory in the presence of closed time like curves. Much more remains to be explored in this direction.

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